Schmidt Norms for Quantum States

Nathaniel Johnston
Collaborative work with David Kribs

University of Guelph

October and November, 2009

Notation

- $\mathcal{H}$ is a (finite-dimensional) Hilbert space. $\mathcal{H}_n$ is an n-dimensional Hilbert space.

Notation

- $\mathcal{H}$ is a (finite-dimensional) Hilbert space. $\mathcal{H}_n$ is an n-dimensional Hilbert space.
- $\mathcal{L}(\mathcal{H})$ is the space of linear operators acting on $\mathcal{H}$.

Notation

- $\mathcal{H}$ is a (finite-dimensional) Hilbert space. $\mathcal{H}_n$ is an n-dimensional Hilbert space.
- $\mathcal{L}(\mathcal{H})$ is the space of linear operators acting on $\mathcal{H}$.
- id_n represents the identity map on $\mathcal{L}(\mathcal{H}_n)$.

Notation

- $\mathcal{H}$ is a (finite-dimensional) Hilbert space. $\mathcal{H}_n$ is an n-dimensional Hilbert space.
- $\mathcal{L}(\mathcal{H})$ is the space of linear operators acting on $\mathcal{H}$.
- id_n represents the identity map on $\mathcal{L}(\mathcal{H}_n)$.
- $|v\rangle \in \mathcal{H}$ is a unit (column) vector represented using Dirac notation. $\langle v| := |v\rangle^*$ is the dual (row) vector. The associated projection $|v\rangle\langle v|$ represents a pure state.

Notation

- $\mathcal{H}$ is a (finite-dimensional) Hilbert space. $\mathcal{H}_n$ is an n-dimensional Hilbert space.
- $\mathcal{L}(\mathcal{H})$ is the space of linear operators acting on $\mathcal{H}$.
- id_n represents the identity map on $\mathcal{L}(\mathcal{H}_n)$.
- $|v\rangle \in \mathcal{H}$ is a unit (column) vector represented using Dirac notation. $\langle v| := |v\rangle^*$ is the dual (row) vector. The associated projection $|v\rangle\langle v|$ represents a pure state.
- $|e_i\rangle$ denotes the i^{th} computational basis vector. $E := \frac{1}{n}\sum_{i,j=1}^{n} |e_i\rangle\langle e_j| \otimes |e_i\rangle\langle e_j|$ is a rank-1 projection onto a maximally-entangled pure state.

Linear Maps on Operators

A linear map $\Phi : \mathcal{L}(\mathcal{H}_n) \rightarrow \mathcal{L}(\mathcal{H}_m)$ (we will assume for convenience from now on that $m \leq n$) is said to be:

Linear Maps on Operators

A linear map $\Phi : \mathcal{L}(\mathcal{H}_n) \to \mathcal{L}(\mathcal{H}_m)$ (we will assume for convenience from now on that $m \leq n$) is said to be:

- **Hermicity-preserving** if $\Phi(X)^* = \Phi(X)$ whenever $X^* = X$.

Linear Maps on Operators

A linear map $\Phi : \mathcal{L}(\mathcal{H}_n) \to \mathcal{L}(\mathcal{H}_m)$ (we will assume for convenience from now on that $m \leq n$) is said to be:

- **Hermicity-preserving** if $\Phi(X)^* = \Phi(X)$ whenever $X^* = X$.
- **Positive** if $\Phi(X) \geq 0$ whenever $X \geq 0$.

Linear Maps on Operators

A linear map $\Phi : \mathcal{L}(\mathcal{H}_n) \rightarrow \mathcal{L}(\mathcal{H}_m)$ (we will assume for convenience from now on that $m \leq n$) is said to be:

- **Hermicity-preserving** if $\Phi(X)^* = \Phi(X)$ whenever $X^* = X$.
- **Positive** if $\Phi(X) \geq 0$ whenever $X \geq 0$.
- k**-positive** if $(id_k \otimes \Phi)(X) \geq 0$ whenever $X \geq 0$.

Linear Maps on Operators

A linear map $\Phi : \mathcal{L}(\mathcal{H}_n) \rightarrow \mathcal{L}(\mathcal{H}_m)$ (we will assume for convenience from now on that $m \leq n$) is said to be:

- **Hermicity-preserving** if $\Phi(X)^* = \Phi(X)$ whenever $X^* = X$.
- **Positive** if $\Phi(X) \geq 0$ whenever $X \geq 0$.
- ***k*-positive** if $(id_k \otimes \Phi)(X) \geq 0$ whenever $X \geq 0$.
- **Completely positive** if Φ is k-positive for all $k \in \mathbb{N}$.

The Choi-Jamiolkowski Isomorphism

We associate any linear map $\Phi : \mathcal{L}(\mathcal{H}_n) \rightarrow \mathcal{L}(\mathcal{H}_m)$ with the operator $(id_n \otimes \Phi)(E) \in \mathcal{L}(\mathcal{H}_n) \otimes \mathcal{L}(\mathcal{H}_m)$. This actually defines an isomorphism known as the **Choi-Jamiolkowski isomorphism**. Φ is:

The Choi-Jamiolkowski Isomorphism

We associate any linear map $\Phi : \mathcal{L}(\mathcal{H}_n) \rightarrow \mathcal{L}(\mathcal{H}_m)$ with the operator $(id_n \otimes \Phi)(E) \in \mathcal{L}(\mathcal{H}_n) \otimes \mathcal{L}(\mathcal{H}_m)$. This actually defines an isomorphism known as the **Choi-Jamiolkowski isomorphism**. Φ is:

- **Hermicity-preserving** if and only if $(id_n \otimes \Phi)(E)$ is Hermitian.

The Choi-Jamiolkowski Isomorphism

We associate any linear map $\Phi : \mathcal{L}(\mathcal{H}_n) \rightarrow \mathcal{L}(\mathcal{H}_m)$ with the operator $(id_n \otimes \Phi)(E) \in \mathcal{L}(\mathcal{H}_n) \otimes \mathcal{L}(\mathcal{H}_m)$. This actually defines an isomorphism known as the **Choi-Jamiolkowski isomorphism**. Φ is:

- **Hermicity-preserving** if and only if $(id_n \otimes \Phi)(E)$ is Hermitian.
- **Positive** if and only if $(\langle a| \otimes \langle b|)(id_n \otimes \Phi)(E)(|a\rangle \otimes |b\rangle) \geq 0$ for all $|a\rangle \in \mathcal{H}_n, |b\rangle \in \mathcal{H}_m$.

The Choi-Jamiolkowski Isomorphism

We associate any linear map $\Phi : \mathcal{L}(\mathcal{H}_n) \to \mathcal{L}(\mathcal{H}_m)$ with the operator $(id_n \otimes \Phi)(E) \in \mathcal{L}(\mathcal{H}_n) \otimes \mathcal{L}(\mathcal{H}_m)$. This actually defines an isomorphism known as the **Choi-Jamiolkowski isomorphism**. Φ is:

- **Hermicity-preserving** if and only if $(id_n \otimes \Phi)(E)$ is Hermitian.
- **Positive** if and only if $(\langle a| \otimes \langle b|)(id_n \otimes \Phi)(E)(|a\rangle \otimes |b\rangle) \geq 0$ for all $|a\rangle \in \mathcal{H}_n, |b\rangle \in \mathcal{H}_m$.
- **Completely positive** if $(id_n \otimes \Phi)(E)$ is positive semidefinite (a famous result of Choi).

Schmidt Rank

Recall the Schmidt decomposition theorem, which states that any vector $|v\rangle \in \mathcal{H}_n \otimes \mathcal{H}_m$ can be written in the following form:

$$|v\rangle = \sum_{i=1}^{r} \alpha_i |a_i\rangle \otimes |b_i\rangle,$$

where each α_i is a non-negative real constant and $\{|a_i\rangle\} \subseteq \mathcal{H}_n$ and $\{|b_i\rangle\} \subseteq \mathcal{H}_m$ are orthonormal sets of vectors.

Schmidt Rank

Recall the Schmidt decomposition theorem, which states that any vector $|v\rangle \in \mathcal{H}_n \otimes \mathcal{H}_m$ can be written in the following form:

$$|v\rangle = \sum_{i=1}^{r} \alpha_i |a_i\rangle \otimes |b_i\rangle,$$

where each α_i is a non-negative real constant and $\{|a_i\rangle\} \subseteq \mathcal{H}_n$ and $\{|b_i\rangle\} \subseteq \mathcal{H}_m$ are orthonormal sets of vectors.

- The least possible r is known as the **Schmidt rank** of $|v\rangle$, denoted $SR(|v\rangle)$.

Schmidt Rank

- The proof of the Schmidt decomposition theorem is just the singular value decomposition is disguise.

Schmidt Rank

- The proof of the Schmidt decomposition theorem is just the singular value decomposition is disguise.
- The Schmidt rank of a vector is easy to compute (takes $O(n^3)$ time).

Schmidt Rank

- The proof of the Schmidt decomposition theorem is just the singular value decomposition is disguise.
- The Schmidt rank of a vector is easy to compute (takes $O(n^3)$ time).
- $SR(|v\rangle) = 1$ if and only if $|v\rangle$ is separable. Conversely, vectors with high Schmidt rank (i.e., $SR(|v\rangle) \approx m$) are considered "highly entangled".

Schmidt Rank

- The proof of the Schmidt decomposition theorem is just the singular value decomposition is disguise.
- The Schmidt rank of a vector is easy to compute (takes $O(n^3)$ time).
- $SR(|v\rangle) = 1$ if and only if $|v\rangle$ is separable. Conversely, vectors with high Schmidt rank (i.e., $SR(|v\rangle) \approx m$) are considered "highly entangled".
- A result of Cubitt, Montanaro and Winter says that the maximum dimension of a subspace $\mathcal{S} \subseteq \mathcal{H}_n \otimes \mathcal{H}_m$ such that $SR(|v\rangle) \geq k$ for all $|v\rangle \in \mathcal{S}$ is given by $(n-k+1)(m-k+1)$.

k-Block Positive Operators

An operator $X \in \mathcal{L}(\mathcal{H}_n) \otimes \mathcal{L}(\mathcal{H}_m)$ is said to be **k-block positive** if

$$\langle v|X|v\rangle \geq 0 \quad \forall\, |v\rangle \text{ with } SR(|v\rangle) \leq k.$$

k-Block Positive Operators

An operator $X \in \mathcal{L}(\mathcal{H}_n) \otimes \mathcal{L}(\mathcal{H}_m)$ is said to be **k-block positive** if

$$\langle v|X|v\rangle \geq 0 \quad \forall\,|v\rangle \text{ with } SR(|v\rangle) \leq k.$$

- The m-block positive operators are exactly the positive operators, since $SR(|v\rangle) \leq m$ for all vectors $|v\rangle$.

k-Block Positive Operators

An operator $X \in \mathcal{L}(\mathcal{H}_n) \otimes \mathcal{L}(\mathcal{H}_m)$ is said to be **k-block positive** if

$$\langle v|X|v\rangle \geq 0 \quad \forall\, |v\rangle \text{ with } SR(|v\rangle) \leq k.$$

- The m-block positive operators are exactly the positive operators, since $SR(|v\rangle) \leq m$ for all vectors $|v\rangle$.
- X is 1-block positive if and only if the linear map associated with it (via the Choi-Jamiolkowski Isomorphism) is positive.

k-Block Positive Operators

An operator $X \in \mathcal{L}(\mathcal{H}_n) \otimes \mathcal{L}(\mathcal{H}_m)$ is said to be **k-block positive** if

$$\langle v|X|v\rangle \geq 0 \quad \forall\, |v\rangle \text{ with } SR(|v\rangle) \leq k.$$

- The m-block positive operators are exactly the positive operators, since $SR(|v\rangle) \leq m$ for all vectors $|v\rangle$.
- X is 1-block positive if and only if the linear map associated with it (via the Choi-Jamiolkowski Isomorphism) is positive.
- X is k-block positive if and only if the linear map associated to it (via the Choi-Jamiolkowski Isomorphism) is k-positive.

Schmidt Number

The Schmidt rank provides a simple measure of entanglement for vectors (pure states). The **Schmidt number** is a generalization of the Schmidt rank to arbitrary density operators (mixed states).

Schmidt Number

The Schmidt rank provides a simple measure of entanglement for vectors (pure states). The **Schmidt number** is a generalization of the Schmidt rank to arbitrary density operators (mixed states).

Any density operator $\rho \in \mathcal{L}(\mathcal{H})$ can be written as

$$\rho = \sum_i p_i |v_i\rangle\langle v_i|,$$

where $\{p_i\}$ forms a probability distribution.

Schmidt Number

The Schmidt rank provides a simple measure of entanglement for vectors (pure states). The **Schmidt number** is a generalization of the Schmidt rank to arbitrary density operators (mixed states).

Any density operator $\rho \in \mathcal{L}(\mathcal{H})$ can be written as

$$\rho = \sum_i p_i |v_i\rangle\langle v_i|,$$

where $\{p_i\}$ forms a probability distribution.

The above decomposition of ρ is not unique. The **Schmidt number** of ρ, denoted $SN(\rho)$, is defined to be the least k such that $SR(|v_i\rangle) \leq k$ for all i, where the $|v_i\rangle$'s form a decomposition of ρ as above.

Schmidt Number

- For any vector $|v\rangle$, $SR(|v\rangle) = SN(|v\rangle\langle v|)$.

Schmidt Number

- For any vector $|v\rangle$, $SR(|v\rangle) = SN(|v\rangle\langle v|)$.
- $SN(\rho) = 1$ if and only if ρ is separable. Conversely, density operators with high Schmidt number (i.e., $SN(\rho) \approx m$) are considered "highly entangled".

Schmidt Number

- For any vector $|v\rangle$, $SR(|v\rangle) = SN(|v\rangle\langle v|)$.
- $SN(\rho) = 1$ if and only if ρ is separable. Conversely, density operators with high Schmidt number (i.e., $SN(\rho) \approx m$) are considered "highly entangled".
- There is a natural duality between operators ρ with $SN(\rho) \leq k$ and k-block positive operators.

Schmidt Number

- For any vector $|v\rangle$, $SR(|v\rangle) = SN(|v\rangle\langle v|)$.
- $SN(\rho) = 1$ if and only if ρ is separable. Conversely, density operators with high Schmidt number (i.e., $SN(\rho) \approx m$) are considered "highly entangled".
- There is a natural duality between operators ρ with $SN(\rho) \leq k$ and k-block positive operators.
- The Schmidt number of an arbitrary density operator is *very* difficult to compute. Not only is it NP-HARD, but we don't even know how to do it yet. Even determining whether or not $SN(\rho) = 1$ is a very active area of research right now.

Schmidt Vector Norms

Let $|v\rangle \in \mathcal{H}_n \otimes \mathcal{H}_m$ and let $1 \leq k \leq m$. Then we define the **Schmidt vector k-norm** of $|v\rangle$ by

$$\big\| |v\rangle \big\|_{s(k)} := \sup_{|w\rangle} \Big\{ \big| \langle w|v\rangle \big| : SR(|w\rangle) \leq k \Big\}.$$

Schmidt Vector Norms

Let $|v\rangle \in \mathcal{H}_n \otimes \mathcal{H}_m$ and let $1 \leq k \leq m$. Then we define the **Schmidt vector k-norm** of $|v\rangle$ by

$$\big\||v\rangle\big\|_{s(k)} := \sup_{|w\rangle} \Big\{ \big|\langle w|v\rangle\big| : SR(|w\rangle) \leq k \Big\}.$$

- Yes, these are actually norms (not hard to check).

Schmidt Vector Norms

Let $|v\rangle \in \mathcal{H}_n \otimes \mathcal{H}_m$ and let $1 \leq k \leq m$. Then we define the **Schmidt vector k-norm** of $|v\rangle$ by

$$\big\| |v\rangle \big\|_{s(k)} := \sup_{|w\rangle} \Big\{ \big| \langle w|v\rangle \big| : SR(|w\rangle) \leq k \Big\}.$$

- Yes, these are actually norms (not hard to check).
- If $k = m$, then this is just the standard Euclidean norm. That is, $\big\| |v\rangle \big\|_{s(m)} = \big\| |v\rangle \big\|$.

Schmidt Vector Norms

Let $|v\rangle \in \mathcal{H}_n \otimes \mathcal{H}_m$ and let $1 \leq k \leq m$. Then we define the **Schmidt vector k-norm** of $|v\rangle$ by

$$\big\| |v\rangle \big\|_{s(k)} := \sup_{|w\rangle} \Big\{ \big| \langle w|v\rangle \big| : SR(|w\rangle) \leq k \Big\}.$$

- Yes, these are actually norms (not hard to check).
- If $k = m$, then this is just the standard Euclidean norm. That is, $\big\| |v\rangle \big\|_{s(m)} = \big\| |v\rangle \big\|$.
- $\big\| |v\rangle \big\|_{s(1)} \leq \big\| |v\rangle \big\|_{s(2)} \leq \cdots \leq \big\| |v\rangle \big\|_{s(m-1)} \leq \big\| |v\rangle \big\|$

Computation

From now on, when we refer to the Schmidt coefficients $\{\alpha_i\}$ of a vector, we will assume that they are written in non-increasing order (i.e., $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_m \geq 0$).

Computation

From now on, when we refer to the Schmidt coefficients $\{\alpha_i\}$ of a vector, we will assume that they are written in non-increasing order (i.e., $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_m \geq 0$).

The Schmidt vector norms are actually very simple to compute, as the following theorem demonstrates:

Theorem

Let $|v\rangle \in \mathcal{H}_n \otimes \mathcal{H}_m$ *have Schmidt coefficients* $\{\alpha_i\}$. *Then*

$$\big\||v\rangle\big\|_{s(k)} = \sqrt{\sum_{i=1}^{k} \alpha_i^2}.$$

Computation

- The Schmidt vector norms are thus easy to compute – it takes $O(n^3)$ time (and it's simple to code in software such as MATLAB).

Computation

- The Schmidt vector norms are thus easy to compute – it takes $O(n^3)$ time (and it's simple to code in software such as MATLAB).
- It is a simple corollary that if $h \leq k$ then $\big\||v\rangle\big\|_{s(h)} \leq \big\||v\rangle\big\|_{s(k)} \leq \sqrt{\frac{k}{h}}\big\||v\rangle\big\|_{s(h)}$.

Computation

- The Schmidt vector norms are thus easy to compute – it takes $O(n^3)$ time (and it's simple to code in software such as MATLAB).
- It is a simple corollary that if $h \leq k$ then $\big\||v\rangle\big\|_{s(h)} \leq \big\||v\rangle\big\|_{s(k)} \leq \sqrt{\frac{k}{h}}\big\||v\rangle\big\|_{s(h)}$.
- $\big\||v\rangle\big\|_{s(h)} = \big\||v\rangle\big\|_{s(k)}$ if and only if $SR(|v\rangle) \leq h$.

Computation

- The Schmidt vector norms are thus easy to compute – it takes $O(n^3)$ time (and it's simple to code in software such as MATLAB).
- It is a simple corollary that if $h \leq k$ then $\big\||v\rangle\big\|_{s(h)} \leq \big\||v\rangle\big\|_{s(k)} \leq \sqrt{\frac{k}{h}}\big\||v\rangle\big\|_{s(h)}$.
- $\big\||v\rangle\big\|_{s(h)} = \big\||v\rangle\big\|_{s(k)}$ if and only if $SR(|v\rangle) \leq h$.
- $\big\||v\rangle\big\|_{s(k)} = \sqrt{\frac{k}{h}}\big\||v\rangle\big\|_{s(h)}$ if and only if $\alpha_1 = \alpha_2 = \cdots = \alpha_k$.

Schmidt Operator Norms

Let $X \in \mathcal{L}(\mathcal{H}_n) \otimes \mathcal{L}(\mathcal{H}_m)$ and let $1 \leq k \leq m$. Then we define the **Schmidt operator k-norm** of X by

$$\big\|X\big\|_{S(k)} := \sup_{|v\rangle,|w\rangle} \Big\{ \big|\langle w|X|v\rangle\big| : SR(|v\rangle), SR(|w\rangle) \leq k \Big\}.$$

Schmidt Operator Norms

Let $X \in \mathcal{L}(\mathcal{H}_n) \otimes \mathcal{L}(\mathcal{H}_m)$ and let $1 \leq k \leq m$. Then we define the **Schmidt operator k-norm** of X by

$$\big\|X\big\|_{S(k)} := \sup_{|v\rangle,|w\rangle} \Big\{ \big| \langle w|X|v\rangle \big| : SR(|v\rangle), SR(|w\rangle) \leq k \Big\}.$$

- Yes, these too are actually norms.

Schmidt Operator Norms

Let $X \in \mathcal{L}(\mathcal{H}_n) \otimes \mathcal{L}(\mathcal{H}_m)$ and let $1 \leq k \leq m$. Then we define the **Schmidt operator k-norm** of X by

$$\big\|X\big\|_{S(k)} := \sup_{|v\rangle,|w\rangle} \Big\{ \big|\langle w|X|v\rangle\big| : SR(|v\rangle), SR(|w\rangle) \leq k \Big\}.$$

- Yes, these too are actually norms.
- If $k = m$, then this is the standard operator norm. That is, $\big\|X\big\|_{S(m)} = \big\|X\big\|$.

Schmidt Operator Norms

Let $X \in \mathcal{L}(\mathcal{H}_n) \otimes \mathcal{L}(\mathcal{H}_m)$ and let $1 \leq k \leq m$. Then we define the **Schmidt operator k-norm** of X by

$$\big\|X\big\|_{S(k)} := \sup_{|v\rangle,|w\rangle} \Big\{ \big|\langle w|X|v\rangle\big| : SR(|v\rangle), SR(|w\rangle) \leq k \Big\}.$$

- Yes, these too are actually norms.
- If $k = m$, then this is the standard operator norm. That is, $\big\|X\big\|_{S(m)} = \big\|X\big\|$.
- $\big\|X\big\|_{S(1)} \leq \big\|X\big\|_{S(2)} \leq \cdots \leq \big\|X\big\|_{S(m-1)} \leq \big\|X\big\|$

Rank-1 Computation

We will see that computing Schmidt operator norms in full generality is extremely difficult. Nonetheless, the following proposition shows us how to do it when the operator has rank 1.

Rank-1 Computation

We will see that computing Schmidt operator norms in full generality is extremely difficult. Nonetheless, the following proposition shows us how to do it when the operator has rank 1.

Proposition

Let $X = |w\rangle\langle v| \in \mathcal{L}(\mathcal{H}_n) \otimes \mathcal{L}(\mathcal{H}_m)$ *be a rank-*1 *operator. Then*

$$\big\|X\big\|_{S(k)} = \big\||w\rangle\big\|_{s(k)}\big\||v\rangle\big\|_{s(k)}.$$

Rank-1 Computation

We will see that computing Schmidt operator norms in full generality is extremely difficult. Nonetheless, the following proposition shows us how to do it when the operator has rank 1.

Proposition

Let $X = |w\rangle\langle v| \in \mathcal{L}(\mathcal{H}_n) \otimes \mathcal{L}(\mathcal{H}_m)$ *be a rank-1 operator. Then*

$$\big\|X\big\|_{S(k)} = \big\||w\rangle\big\|_{s(k)}\big\||v\rangle\big\|_{s(k)}.$$

- Because we know how to compute the Schmidt vector norms, we can use this result to compute the Schmidt operator norm of rank-1 operators.

Rank-1 Computation

- The proposition implies that $\big\| |v\rangle\langle v| \big\|_{S(k)} = \big\| |v\rangle \big\|_{s(k)}^2$, so the Schmidt operator norm on density operators generalizes the Schmidt vector norm on vectors, as we would hope.

Rank-1 Computation

- The proposition implies that $\big\| |v\rangle\langle v| \big\|_{S(k)} = \big\| |v\rangle \big\|_{s(k)}^2$, so the Schmidt operator norm on density operators generalizes the Schmidt vector norm on vectors, as we would hope.
- As an example, recall the rank-1 projection operator $E = \frac{1}{n}\sum_{i,j=1}^{n} |e_i\rangle\langle e_j| \otimes |e_i\rangle\langle e_j| \in \mathcal{L}(\mathcal{H}_n) \otimes \mathcal{L}(\mathcal{H}_n)$. Using the proposition, we see that

$$\big\|E\big\|_{S(k)} = \Big\| \sum_{i=1}^{n} \frac{1}{\sqrt{n}} |e_i\rangle \otimes |e_i\rangle \Big\|_{s(k)}^2 = \sum_{i=1}^{k} \left(\frac{1}{\sqrt{n}}\right)^2 = \frac{k}{n}.$$

Normal Operators

Recall that in the definition of the Schmidt operator norms, we take a supremum over two vectors $|v\rangle$ and $|w\rangle$. The following proposition shows if X is normal then it is enough to take the supremum only over $|v\rangle$:

Normal Operators

Recall that in the definition of the Schmidt operator norms, we take a supremum over two vectors $|v\rangle$ and $|w\rangle$. The following proposition shows if X is normal then it is enough to take the supremum only over $|v\rangle$:

Proposition

Let $X \in \mathcal{L}(\mathcal{H}_n) \otimes \mathcal{L}(\mathcal{H}_m)$ be normal. Then

$$\big\|X\big\|_{S(k)} = \sup_{|v\rangle} \Big\{ \big|\langle v|X|v\rangle\big| : SR(|v\rangle) \leq k \Big\}.$$

Normal Operators

Recall that in the definition of the Schmidt operator norms, we take a supremum over two vectors $|v\rangle$ and $|w\rangle$. The following proposition shows if X is normal then it is enough to take the supremum only over $|v\rangle$:

Proposition

Let $X \in \mathcal{L}(\mathcal{H}_n) \otimes \mathcal{L}(\mathcal{H}_m)$ be normal. Then

$$\big\|X\big\|_{S(k)} = \sup_{|v\rangle} \Big\{ \big|\langle v|X|v\rangle\big| : SR(|v\rangle) \leq k \Big\}.$$

- In the $k = m$ case, this result captures a well-known result about the operator norm for normal operators.

Normal Operators

- In the $k = 1$ case, it follows that if X is normal then $\left\|X\right\|_{S(1)}$ coincides with the recently-studied *local numerical radius* of Gawron et. al.

Normal Operators

- In the $k = 1$ case, it follows that if X is normal then $\big\|X\big\|_{S(1)}$ coincides with the recently-studied *local numerical radius* of Gawron et. al.
- A simple corollary of the proposition is that if X is normal, then

$$\big\|X\big\|_{S(k)} = \sup_{\rho} \Big\{ |\mathrm{Tr}(X\rho)| : SN(\rho) \leq k \Big\}.$$

Connection with k-Block Positivity

The following result shows an intimate connection between the Schmidt operator norms and k-block positivity of an operator.

Connection with k-Block Positivity

The following result shows an intimate connection between the Schmidt operator norms and k-block positivity of an operator.

Proposition

Let $0 \leq X \in \mathcal{L}(\mathcal{H}_n) \otimes \mathcal{L}(\mathcal{H}_m)$ *and let* $c \in \mathbb{R}$. *Then* $cI - X$ *is* k*-block positive if and only if* $c \geq \big\|X\big\|_{S(k)}$.

Connection with k-Block Positivity

The following result shows an intimate connection between the Schmidt operator norms and k-block positivity of an operator.

Proposition

Let $0 \leq X \in \mathcal{L}(\mathcal{H}_n) \otimes \mathcal{L}(\mathcal{H}_m)$ *and let* $c \in \mathbb{R}$*. Then* $cI - X$ *is* k*-block positive if and only if* $c \geq \big\|X\big\|_{S(k)}$.

- This shows that computing Schmidt operator norms for positive operators is equivalent to the problem of determining k-block positivity of Hermitian operators.

Connection with k-Block Positivity

The following result shows an intimate connection between the Schmidt operator norms and k-block positivity of an operator.

Proposition

Let $0 \leq X \in \mathcal{L}(\mathcal{H}_n) \otimes \mathcal{L}(\mathcal{H}_m)$ and let $c \in \mathbb{R}$. Then $cI - X$ is k-block positive if and only if $c \geq \big\|X\big\|_{S(k)}$.

- This shows that computing Schmidt operator norms for positive operators is equivalent to the problem of determining k-block positivity of Hermitian operators.
- Determining k-block positivity of Hermitian operators has been extensively studied, but no complete method is known. This suggests that computing Schmidt operator norms is a very difficult problem.

Computable Upper Bound

Since we likely can't compute the Schmidt operator norms in full generality, we will now focus on finding inequalities to bound them.

Computable Upper Bound

Since we likely can't compute the Schmidt operator norms in full generality, we will now focus on finding inequalities to bound them.

Proposition

Let $X \in \mathcal{L}(\mathcal{H}_n) \otimes \mathcal{L}(\mathcal{H}_m)$ be normal with eigenvalues $\{\lambda_i\}$ and corresponding eigenvectors $\{|v_i\rangle\}$. Then

$$\big\|X\big\|_{S(k)} \leq \sum_i |\lambda_i| \big\||v_i\rangle\big\|_{s(k)}^2.$$

Computable Upper Bound

Since we likely can't compute the Schmidt operator norms in full generality, we will now focus on finding inequalities to bound them.

Proposition

Let $X \in \mathcal{L}(\mathcal{H}_n) \otimes \mathcal{L}(\mathcal{H}_m)$ be normal with eigenvalues $\{\lambda_i\}$ and corresponding eigenvectors $\{|v_i\rangle\}$. Then

$$\big\|X\big\|_{S(k)} \leq \sum_i |\lambda_i| \big\||v_i\rangle\big\|_{s(k)}^2.$$

- The result follows easily from the Spectral decomposition theorem.

Computable Upper Bound

We can similarly derive an upper bound for non-normal operators by recalling that any operator X can be written as $X = \sum_i a_i|v_i\rangle\langle w_i|$. Then:

Computable Upper Bound

We can similarly derive an upper bound for non-normal operators by recalling that any operator X can be written as $X = \sum_i a_i|v_i\rangle\langle w_i|$. Then:

$$\big\|X\big\|_{S(k)} \leq \sum_i |a_i| \big\||v_i\rangle\big\|_{s(k)} \big\||w_i\rangle\big\|_{s(k)}.$$

Computable Upper Bound

We can similarly derive an upper bound for non-normal operators by recalling that any operator X can be written as $X = \sum_i a_i|v_i\rangle\langle w_i|$. Then:

$$\big\|X\big\|_{S(k)} \leq \sum_i |a_i|\big\||v_i\rangle\big\|_{s(k)}\big\||w_i\rangle\big\|_{s(k)}.$$

- The decomposition of X can be chosen so that the number of terms in the sum is $\mathrm{rank}(X)$. Thus, we saw earlier that equality is attained in the rank-1 case.

Equivalence of Schmidt Operator Norms

Because $\mathcal{L}(\mathcal{H}_n) \otimes \mathcal{L}(\mathcal{H}_m)$ is finite-dimensional, we know that the Schmidt operator norms must all be equivalent. The following theorem quantifies this fact.

Equivalence of Schmidt Operator Norms

Because $\mathcal{L}(\mathcal{H}_n) \otimes \mathcal{L}(\mathcal{H}_m)$ is finite-dimensional, we know that the Schmidt operator norms must all be equivalent. The following theorem quantifies this fact.

Theorem

Let $X \in \mathcal{L}(\mathcal{H}_n) \otimes \mathcal{L}(\mathcal{H}_m)$ and suppose $h \leq k$. Then

$$\|X\|_{S(h)} \leq \|X\|_{S(k)} \leq \frac{k}{h}\|X\|_{S(h)}.$$

Equivalence of Schmidt Operator Norms

Because $\mathcal{L}(\mathcal{H}_n) \otimes \mathcal{L}(\mathcal{H}_m)$ is finite-dimensional, we know that the Schmidt operator norms must all be equivalent. The following theorem quantifies this fact.

Theorem

Let $X \in \mathcal{L}(\mathcal{H}_n) \otimes \mathcal{L}(\mathcal{H}_m)$ and suppose $h \leq k$. Then

$$\big\|X\big\|_{S(h)} \leq \big\|X\big\|_{S(k)} \leq \frac{k}{h}\big\|X\big\|_{S(h)}.$$

- These equivalence inequalities are tight. To see this, recall from earlier than $\big\|E\big\|_{S(k)} = \frac{k}{n}$.

Equivalence of Schmidt Operator Norms

- Unlike the equivalence result for Schmidt vector norms, it is not easy to find the conditions under which equality is attained in either of the Schmidt operator norm equivalence inequalities.

Equivalence of Schmidt Operator Norms

- Unlike the equivalence result for Schmidt vector norms, it is not easy to find the conditions under which equality is attained in either of the Schmidt operator norm equivalence inequalities.
- In the case of an orthogonal projection $P = P^2 = P^*$, we can improve the lower bound as follows:

$$\big\|P\big\|_{S(h)} + \frac{k-h}{m-1}\left(1 - \big\|P\big\|_{S(h)}\right) \leq \big\|P\big\|_{S(k)}$$

Inequalities for Projections

We will see later that the Schmidt operator norms of projections are particularly important, so we provide two lower bounds here that only apply to orthogonal projections $P = P^2 = P^*$.

Inequalities for Projections

We will see later that the Schmidt operator norms of projections are particularly important, so we provide two lower bounds here that only apply to orthogonal projections $P = P^2 = P^*$.

$$\big\|P\big\|_{S(k)} \geq \min\left\{1, \frac{k}{\left\lceil \frac{1}{2}\left(n + m - \sqrt{(n-m)^2 + 4\mathrm{rank}(P) - 4}\right)\right\rceil}\right\}.$$

Inequalities for Projections

We will see later that the Schmidt operator norms of projections are particularly important, so we provide two lower bounds here that only apply to orthogonal projections $P = P^2 = P^*$.

$$\big\|P\big\|_{S(k)} \geq \min\left\{1, \frac{k}{\left\lceil \frac{1}{2}\left(n + m - \sqrt{(n-m)^2 + 4\mathrm{rank}(P) - 4}\right)\right\rceil}\right\}.$$

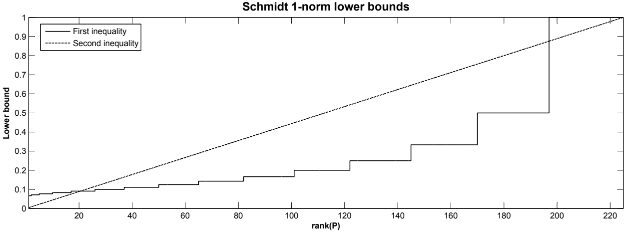
$$\big\|P\big\|_{S(k)} \geq \frac{(k-1)mn + (m-k)\mathrm{rank}(P)}{mn(m-1)}.$$

Inequalities for Projections

The first inequality is best for very low-rank or high-rank projections, but the second inequality is best for moderate-rank projections – this is demonstrated by the following image.

Inequalities for Projections

The first inequality is best for very low-rank or high-rank projections, but the second inequality is best for moderate-rank projections – this is demonstrated by the following image.



A comparison of the two lower bounds for the Schmidt norm on projections in the $n = m = 15$ and $k = 1$ case.

Introduction to Semidefinite Programs

Assume we have a Hermicity-preserving linear map $\Phi : \mathcal{L}(\mathcal{H}_n) \rightarrow \mathcal{L}(\mathcal{H}_m)$ and two operators $A \in \mathcal{L}(\mathcal{H}_n)$ and $B \in \mathcal{L}(\mathcal{H}_m)$. Then the corresponding semidefinite program is given by the following pair of optimization problems:

Introduction to Semidefinite Programs

Assume we have a Hermicity-preserving linear map $\Phi : \mathcal{L}(\mathcal{H}_n) \rightarrow \mathcal{L}(\mathcal{H}_m)$ and two operators $A \in \mathcal{L}(\mathcal{H}_n)$ and $B \in \mathcal{L}(\mathcal{H}_m)$. Then the corresponding semidefinite program is given by the following pair of optimization problems:

Primal problem

$$\begin{aligned} \text{maximize:} \quad & \mathrm{Tr}(AX) \\ \text{subject to:} \quad & \Phi(X) \leq B \\ & X \geq 0 \end{aligned}$$

Dual problem

$$\begin{aligned} \text{minimize:} \quad & \mathrm{Tr}(BY) \\ \text{subject to:} \quad & \Phi^{\dagger}(Y) \geq A \\ & Y \geq 0 \end{aligned}$$

Introduction to Semidefinite Programs

Methods are known for solving semidefinite programs to any desired accuracy ϵ in time that is (roughly speaking) polynomial in n, m, and $log(1/\epsilon)$.

Introduction to Semidefinite Programs

Methods are known for solving semidefinite programs to any desired accuracy ϵ in time that is (roughly speaking) polynomial in n, m, and $log(1/\epsilon)$.

- Under certain conditions (which will be satisfied in this talk), the optimal value of the primal problem equals the optimal value of the dual problem.

Introduction to Semidefinite Programs

Methods are known for solving semidefinite programs to any desired accuracy ϵ in time that is (roughly speaking) polynomial in n, m, and $log(1/\epsilon)$.

- Under certain conditions (which will be satisfied in this talk), the optimal value of the primal problem equals the optimal value of the dual problem.
- The duality theory of semidefinite programs can lead to nice theoretical results.

Introduction to Semidefinite Programs

Methods are known for solving semidefinite programs to any desired accuracy ϵ in time that is (roughly speaking) polynomial in n, m, and $log(1/\epsilon)$.

- Under certain conditions (which will be satisfied in this talk), the optimal value of the primal problem equals the optimal value of the dual problem.
- The duality theory of semidefinite programs can lead to nice theoretical results.
- We will use semidefinite programming to upper bound the Schmidt operator norms, and compute them exactly in low dimensions.

Semidefinite Programs for Schmidt Operator Norms

Let $\Phi_k : \mathcal{L}(\mathcal{H}_n) \rightarrow \mathcal{L}(\mathcal{H}_m)$ be a k-positive linear map and let $X \geq 0$. The the following pair of optimization problems is a semidefinite program:

Semidefinite Programs for Schmidt Operator Norms

Let $\Phi_k : \mathcal{L}(\mathcal{H}_n) \rightarrow \mathcal{L}(\mathcal{H}_m)$ be a k-positive linear map and let $X \geq 0$. The the following pair of optimization problems is a semidefinite program:

Primal problem

$$\begin{aligned} \text{max:} \quad & \mathrm{Tr}(X\rho) \\ \text{s.t.:} \quad & (id_n \otimes \Phi_k)(\rho) \geq 0 \\ & \mathrm{Tr}(\rho) = 1 \\ & \rho \geq 0 \end{aligned}$$

Dual problem

$$\begin{aligned} \text{min:} \quad & \lambda \\ \text{s.t.:} \quad & \lambda I_n \otimes I_m \geq (id_n \otimes \Phi_k^\dagger)(Y) + X \\ & Y \geq 0 \end{aligned}$$

Semidefinite Programs for Schmidt Operator Norms

Let $\Phi_k : \mathcal{L}(\mathcal{H}_n) \rightarrow \mathcal{L}(\mathcal{H}_m)$ be a k-positive linear map and let $X \geq 0$. The the following pair of optimization problems is a semidefinite program:

Primal problem

$$\begin{aligned} \text{max:} \quad & \mathrm{Tr}(X\rho) \\ \text{s.t.:} \quad & (id_n \otimes \Phi_k)(\rho) \geq 0 \\ & \mathrm{Tr}(\rho) = 1 \\ & \rho \geq 0 \end{aligned}$$

Dual problem

$$\begin{aligned} \text{min:} \quad & \lambda \\ \text{s.t.:} \quad & \lambda I_n \otimes I_m \geq (id_n \otimes \Phi_k^\dagger)(Y) + X \\ & Y \geq 0 \end{aligned}$$

- In the primal problem, we are essentially maximizing $\mathrm{Tr}(X\rho)$ over all density operators ρ with $SN(\rho) \leq k$.

Semidefinite Programs for Schmidt Operator Norms

- The optimal value for each of these semidefinite programs is an upper bound for $\left\|X\right\|_{S(k)}$.

Semidefinite Programs for Schmidt Operator Norms

- The optimal value for each of these semidefinite programs is an upper bound for $\left\| X \right\|_{S(k)}$.
- There is a particular k-positive Φ_k such that the optimal value of the associated semidefinite program is exactly $\left\| X \right\|_{S(k)}$.

Semidefinite Programs for Schmidt Operator Norms

- The optimal value for each of these semidefinite programs is an upper bound for $\left\|X\right\|_{S(k)}$.
- There is a particular k-positive Φ_k such that the optimal value of the associated semidefinite program is exactly $\left\|X\right\|_{S(k)}$.
- When $m = 2$ and $n \leq 3$, it is enough to consider just the transpose map. That is, the optimal value of the semidefinite program associated with the transpose map always gives exactly $\left\|X\right\|_{S(1)}$ in this case.

Semidefinite Programs for Schmidt Operator Norms

- The optimal value for each of these semidefinite programs is an upper bound for $\|X\|_{S(k)}$.
- There is a particular k-positive Φ_k such that the optimal value of the associated semidefinite program is exactly $\|X\|_{S(k)}$.
- When $m = 2$ and $n \leq 3$, it is enough to consider just the transpose map. That is, the optimal value of the semidefinite program associated with the transpose map always gives exactly $\|X\|_{S(1)}$ in this case.
- This semidefinite programming approach, as well as all of the previous bounds and methods, have been implemented in MATLAB.

Semidefinite Programs for Schmidt Operator Norms

The duality theory of semidefinite programming leads to the following two results:

Semidefinite Programs for Schmidt Operator Norms

The duality theory of semidefinite programming leads to the following two results:

Theorem

Let $X \geq 0$. Then

$$\big\|X\big\|_{S(k)} = \inf_{Y} \big\{\big\|X + Y\big\| : Y \textit{ is k-block positive}\big\}.$$

Semidefinite Programs for Schmidt Operator Norms

The duality theory of semidefinite programming leads to the following two results:

Theorem

Let $X \geq 0$. Then

$$\big\|X\big\|_{S(k)} = \inf_{Y} \big\{ \big\|X + Y\big\| : Y \text{ is } k\text{-block positive} \big\}.$$

Corollary

Let $Y = Y^*$. Then Y is k-block positive if and only if

$$\big\|X\big\|_{S(k)} \leq \big\|X + Y\big\| \quad \forall\, X \geq 0.$$

Testing k-Block Positivity

We will now see that the Schmidt operator norms can be used to compute the k-block positivity of Hermitian operators in many cases.

Testing k-Block Positivity

We will now see that the Schmidt operator norms can be used to compute the k-block positivity of Hermitian operators in many cases.

- This is important for quantum information theory as k-block positive operators can be used to detect the Schmidt number of mixed states via the theory of k-entanglement witnesses.

Testing k-Block Positivity

We will now see that the Schmidt operator norms can be used to compute the k-block positivity of Hermitian operators in many cases.

- This is important for quantum information theory as k-block positive operators can be used to detect the Schmidt number of mixed states via the theory of k-entanglement witnesses.
- Can be used to detect k-positivity of linear maps via the Choi-Jamiolkowski isomorphism.

Testing k-Block Positivity

Throughout this section, we assume that $X = X^*$ and we will make the following notational conveniences:

Testing k-Block Positivity

Throughout this section, we assume that $X = X^*$ and we will make the following notational conveniences:

- $\{\lambda_i^+\}$ are the positive eigenvalues of X, which have corresponding eigenvectors $\{|v_i^+\rangle\}$. $\{\lambda_i^-\}$ are the negative eigenvalues of X, which have corresponding eigenvectors $\{|v_i^-\rangle\}$. The eigenvectors corresponding to the zero eigenvalues are $\{|v_i^0\rangle\}$.

Testing k-Block Positivity

Throughout this section, we assume that $X = X^*$ and we will make the following notational conveniences:

- $\{\lambda_i^+\}$ are the positive eigenvalues of X, which have corresponding eigenvectors $\{|v_i^+\rangle\}$. $\{\lambda_i^-\}$ are the negative eigenvalues of X, which have corresponding eigenvectors $\{|v_i^-\rangle\}$. The eigenvectors corresponding to the zero eigenvalues are $\{|v_i^0\rangle\}$.
- $X^+ := \sum_i \lambda_i^+ |v_i^+\rangle\langle v_i^+| \geq 0$ and $X^- := \sum_i \lambda_i^- |v_i^-\rangle\langle v_i^-| \leq 0$ are defined to be the positive and negative parts of X, respectively.

Testing k-Block Positivity

- $P_X^0 := \sum_i |v_i^0\rangle\langle v_i^0|$ and $P_X^- := \sum_i |v_i^-\rangle\langle v_i^-|$ denote the projections onto the nullspace and negative part of X, respectively.

Testing k-Block Positivity

- $P_X^0 := \sum_i |v_i^0\rangle\langle v_i^0|$ and $P_X^- := \sum_i |v_i^-\rangle\langle v_i^-|$ denote the projections onto the nullspace and negative part of X, respectively.

Theorem

1. *If* $\big\|P_X^-\big\|_{S(k)} = 1$ *then* X *is not* k*-block positive.*
2. *If* $\big\|P_X^0 + P_X^-\big\|_{S(k)} < 1$ *and* $\lambda_i^+ \geq \frac{\|X^-\|_{S(k)}}{1-\|P_X^0+P_X^-\|_{S(k)}}$ *for all* i*, then* X *is* k*-block positive.*
3. *If* $\big\|P_X^-\big\|_{S(k)} < 1$*, all of the negative eigenvalues are equal,* X *is nonsingular, and* $\lambda_i^+ < \frac{\|X^-\|_{S(k)}}{1-\|P_X^-\|_{S(k)}}$ *for all* i*, then* X *is not* k*-block positive.*

Testing k-Block Positivity

The theorem is quite technical and scary-looking. However, it has several corollaries that follow from the various Schmidt operator norm inequalities that we already saw. We will state some of these corollaries now.

Testing k-Block Positivity

The theorem is quite technical and scary-looking. However, it has several corollaries that follow from the various Schmidt operator norm inequalities that we already saw. We will state some of these corollaries now.

Corollary (Kuah and Sudarshan)

Let $\phi_1, \phi_2 : \mathcal{L}(\mathcal{H}_n) \rightarrow \mathcal{L}(\mathcal{H}_m)$ be completely positive linear maps and let $\Phi := \phi_1 - \phi_2$ be a Hermicity-preserving map. Suppose ϕ_2 has canonical (i.e., orthogonal) Kraus operators $\{F_i\}$. If $\mathrm{rank}(F_i) \leq k$ for some i, then Φ is not k-positive.

Testing k-Block Positivity

We can derive some simple tests for k-block positivity based on the negative eigenvalues of X.

Testing k-Block Positivity

We can derive some simple tests for k-block positivity based on the negative eigenvalues of X.

Corollary (number of negative eigenvalues)

Suppose $X = X^* \in \mathcal{L}(\mathcal{H}_n) \otimes \mathcal{L}(\mathcal{H}_m)$ is k-block positive. Then it has at most $(n-k)(m-k)$ negative eigenvalues.

Testing k-Block Positivity

We can derive some simple tests for k-block positivity based on the negative eigenvalues of X.

Corollary (number of negative eigenvalues)

Suppose $X = X^* \in \mathcal{L}(\mathcal{H}_n) \otimes \mathcal{L}(\mathcal{H}_m)$ is k-block positive. Then it has at most $(n-k)(m-k)$ negative eigenvalues.

Corollary (magnitude of negative eigenvalues)

Suppose $X = X^* \in \mathcal{L}(\mathcal{H}_n) \otimes \mathcal{L}(\mathcal{H}_m)$ is k-block positive. Denote the maximal and minimal eigenvalues of X by λ_{max} and λ_{min}, respectively. Then

$$\frac{\lambda_{min}}{\lambda_{max}} \geq 1 - \frac{m}{k}.$$

Testing k-Block Positivity

The previous two corollaries are easy enough to prove using other methods. The following corollary provides a way of "interpolating" between them by using the inequalities of the last section.

Testing k-Block Positivity

The previous two corollaries are easy enough to prove using other methods. The following corollary provides a way of "interpolating" between them by using the inequalities of the last section.

Corollary (number and magnitude of negative eigenvalues)

Suppose $X = X^* \in \mathcal{L}(\mathcal{H}_n) \otimes \mathcal{L}(\mathcal{H}_m)$ is k-block positive with r negative eigenvalues. Denote the maximal and minimal eigenvalues of X by λ_{max} and λ_{min}, respectively. Then

$$\frac{\lambda_{min}}{\lambda_{max}} \geq 1 - \frac{\left\lceil \frac{1}{2}\left(n + m - \sqrt{(n-m)^2 + 4r - 4}\right)\right\rceil}{k} \quad \textit{and}$$

$$\frac{\lambda_{min}}{\lambda_{max}} \geq 1 - \frac{mn(m-1)}{(k-1)mn + (m-k)r}.$$

Testing k-Block Positivity

As one final corollary, we now present a necessary and sufficient condition for an operator with exactly two distinct eigenvalues to be k-block positive.

Testing k-Block Positivity

As one final corollary, we now present a necessary and sufficient condition for an operator with exactly two distinct eigenvalues to be k-block positive.

Corollary

Let $X = X^* \in \mathcal{L}(\mathcal{H}_n) \otimes \mathcal{L}(\mathcal{H}_m)$ have exactly two distinct eigenvalues $\lambda_1 > \lambda_2$. Then X is k-block positive if and only if

$$\|P_X^-\|_{S(k)} \leq \frac{\lambda_1}{\lambda_1 - \lambda_2}.$$

Werner States

Let $T : \mathcal{L}(\mathcal{H}_n) \rightarrow \mathcal{L}(\mathcal{H}_n)$ be the transpose map. Consider the following family of states that are parametrized by a single real variable α:

$$\rho_\alpha := \frac{1}{n^2 - \alpha n}(I - \alpha(id_n \otimes T)(nE)) \in \mathcal{L}(\mathcal{H}_n) \otimes \mathcal{L}(\mathcal{H}_n).$$

Werner States

Let $T : \mathcal{L}(\mathcal{H}_n) \rightarrow \mathcal{L}(\mathcal{H}_n)$ be the transpose map. Consider the following family of states that are parametrized by a single real variable α:

$$\rho_\alpha := \frac{1}{n^2 - \alpha n}(I - \alpha(id_n \otimes T)(nE)) \in \mathcal{L}(\mathcal{H}_n) \otimes \mathcal{L}(\mathcal{H}_n).$$

- Such states are known as **Werner states**.

Werner States

Let $T : \mathcal{L}(\mathcal{H}_n) \rightarrow \mathcal{L}(\mathcal{H}_n)$ be the transpose map. Consider the following family of states that are parametrized by a single real variable α:

$$\rho_\alpha := \frac{1}{n^2 - \alpha n}(I - \alpha(id_n \otimes T)(nE)) \in \mathcal{L}(\mathcal{H}_n) \otimes \mathcal{L}(\mathcal{H}_n).$$

- Such states are known as **Werner states**.
- The operator $(id_n \otimes T)(nE)$ is sometimes referred to as the **swap operator** because it maps $|e_i\rangle \otimes |e_j\rangle$ to $|e_j\rangle \otimes |e_i\rangle$.

Bound Entangled States

A state $\rho \in \mathcal{L}(\mathcal{H}_n) \otimes \mathcal{L}(\mathcal{H}_m)$ is said to be **bound entangled** if $(id_n \otimes T)(\rho)^{\otimes \ell} \in \mathcal{L}(\mathcal{H}_n^{\otimes \ell}) \otimes \mathcal{L}(\mathcal{H}_m^{\otimes \ell})$ is 2-block positive for all $\ell \geq 1$.

Bound Entangled States

A state $\rho \in \mathcal{L}(\mathcal{H}_n) \otimes \mathcal{L}(\mathcal{H}_m)$ is said to be **bound entangled** if $(id_n \otimes T)(\rho)^{\otimes \ell} \in \mathcal{L}(\mathcal{H}_n^{\otimes \ell}) \otimes \mathcal{L}(\mathcal{H}_m^{\otimes \ell})$ is 2-block positive for all $\ell \geq 1$.

- Equivalently, a state is bound entangled if it has zero **distillable entanglement**.

Bound Entangled States

A state $\rho \in \mathcal{L}(\mathcal{H}_n) \otimes \mathcal{L}(\mathcal{H}_m)$ is said to be **bound entangled** if $(id_n \otimes T)(\rho)^{\otimes \ell} \in \mathcal{L}(\mathcal{H}_n^{\otimes \ell}) \otimes \mathcal{L}(\mathcal{H}_m^{\otimes \ell})$ is 2-block positive for all $\ell \geq 1$.

- Equivalently, a state is bound entangled if it has zero **distillable entanglement**.
- This means that bound entangled states are states that can not be transformed via local operations and classical communication into a pure maximally entangled state.

Bound Entangled States

A state $\rho \in \mathcal{L}(\mathcal{H}_n) \otimes \mathcal{L}(\mathcal{H}_m)$ is said to be **bound entangled** if $(id_n \otimes T)(\rho)^{\otimes \ell} \in \mathcal{L}(\mathcal{H}_n^{\otimes \ell}) \otimes \mathcal{L}(\mathcal{H}_m^{\otimes \ell})$ is 2-block positive for all $\ell \geq 1$.

- Equivalently, a state is bound entangled if it has zero **distillable entanglement**.
- This means that bound entangled states are states that can not be transformed via local operations and classical communication into a pure maximally entangled state.
- Separable states are clearly bound entangled.

Bound Entangled States

- Under the definition given on the previous slide, it is also clear that **positive partial transpose (PPT)** states are bound entangled. That is, if $\rho \geq 0$ and $(id_n \otimes T)(\rho) \geq 0$ then ρ is bound entangled.

Bound Entangled States

- Under the definition given on the previous slide, it is also clear that **positive partial transpose (PPT)** states are bound entangled. That is, if $\rho \geq 0$ and $(id_n \otimes T)(\rho) \geq 0$ then ρ is bound entangled.
- It has been an open question for over a decade whether or not other bound entangled states exist.

Bound Entangled States

- Under the definition given on the previous slide, it is also clear that **positive partial transpose (PPT)** states are bound entangled. That is, if $\rho \geq 0$ and $(id_n \otimes T)(\rho) \geq 0$ then ρ is bound entangled.
- It has been an open question for over a decade whether or not other bound entangled states exist.
- That is, does there exist a bound entangled state ρ with non-positive partial transpose (NPPT)?

Bound Entangled States

- Mathematically, does there exist a state ρ such that $(id_n \otimes T)(\rho) \not\geq 0$ yet $(id_n \otimes T)(\rho)^{\otimes \ell}$ is 2-block positive for all $\ell \geq 1$?

Bound Entangled States

- Mathematically, does there exist a state ρ such that $(id_n \otimes T)(\rho) \not\geq 0$ yet $(id_n \otimes T)(\rho)^{\otimes \ell}$ is 2-block positive for all $\ell \geq 1$?
- Put yet another way, does there exist a completely positive linear map Φ such that $T \circ \Phi$ is not completely positive, yet $(T \circ \Phi)^{\otimes \ell}$ is 2-positive for every $\ell \geq 1$?

Bound Entangled States

- Mathematically, does there exist a state ρ such that $(id_n \otimes T)(\rho) \not\geq 0$ yet $(id_n \otimes T)(\rho)^{\otimes \ell}$ is 2-block positive for all $\ell \geq 1$?
- Put yet another way, does there exist a completely positive linear map Φ such that $T \circ \Phi$ is not completely positive, yet $(T \circ \Phi)^{\otimes \ell}$ is 2-positive for every $\ell \geq 1$?
- It has been shown that it is enough to consider Werner states – that is, there exist NPPT bound entangled states if and only if there exist NPPT Werner states.

Bound Entangled Werner States

Recall that a Werner state ρ_α is (ignoring the scaling factor in front) of the form

$$\rho_\alpha = I - \alpha(id_n \otimes T)(nE).$$

Bound Entangled Werner States

Recall that a Werner state ρ_α is (ignoring the scaling factor in front) of the form

$$\rho_\alpha = I - \alpha(id_n \otimes T)(nE).$$

- It's easy to see that

$$(id_n \otimes T)(\rho_\alpha) = I - \alpha nE.$$

Bound Entangled Werner States

Recall that a Werner state ρ_α is (ignoring the scaling factor in front) of the form

$$\rho_\alpha = I - \alpha(id_n \otimes T)(nE).$$

- It's easy to see that

$$(id_n \otimes T)(\rho_\alpha) = I - \alpha nE.$$

- But wait a minute – the operator $I - \alpha nE$ has only two distinct eigenvalues. Recall that the Schmidt operator norms gave us a complete characterization of k-positivity of operators with exactly two distinct eigenvalues.

Bound Entangled Werner States

Theorem

Let ρ_α be a Werner state. Then $(id_n \otimes T)(\rho_\alpha)$ is k-block positive if and only if $\alpha \leq \frac{1}{k}$.

Bound Entangled Werner States

Theorem

Let ρ_α be a Werner state. Then $(id_n \otimes T)(\rho_\alpha)$ is k-block positive if and only if $\alpha \leq \frac{1}{k}$.

- Two special cases of this result are well-known. First, ρ_α has positive partial transpose if and only if $\alpha \leq \frac{1}{n}$. Second, ρ_α is 2-positive if and only if $\alpha \leq \frac{1}{2}$.

Bound Entangled Werner States

Theorem

Let ρ_α be a Werner state. Then $(id_n \otimes T)(\rho_\alpha)$ is k-block positive if and only if $\alpha \leq \frac{1}{k}$.

- Two special cases of this result are well-known. First, ρ_α has positive partial transpose if and only if $\alpha \leq \frac{1}{n}$. Second, ρ_α is 2-positive if and only if $\alpha \leq \frac{1}{2}$.
- This shows that the "interesting region" of α's is $\alpha \in (\frac{1}{n}, \frac{1}{2}]$, since ρ_α is PPT for $\alpha \leq \frac{1}{n}$ and $(id_n \otimes T)(\rho_\alpha)$ is not 2-positive (and hence ρ_α is not bound entangled) for $\alpha > \frac{1}{2}$.

Bound Entangled Werner States

Theorem

Let ρ_α be a Werner state. Then $(id_n \otimes T)(\rho_\alpha)$ is k-block positive if and only if $\alpha \leq \frac{1}{k}$.

- Two special cases of this result are well-known. First, ρ_α has positive partial transpose if and only if $\alpha \leq \frac{1}{n}$. Second, ρ_α is 2-positive if and only if $\alpha \leq \frac{1}{2}$.
- This shows that the "interesting region" of α's is $\alpha \in (\frac{1}{n}, \frac{1}{2}]$, since ρ_α is PPT for $\alpha \leq \frac{1}{n}$ and $(id_n \otimes T)(\rho_\alpha)$ is not 2-positive (and hence ρ_α is not bound entangled) for $\alpha > \frac{1}{2}$.
- Determining k-block positivity of $(id_n \otimes T)(\rho_\alpha)^{\otimes \ell}$ for $\ell > 1$ is much more difficult. But we give it a shot...

Bound Entangled Werner States

Define a family of projections $\{P_\ell\}$ of ever-increasing size recursively by

Bound Entangled Werner States

Define a family of projections $\{P_\ell\}$ of ever-increasing size recursively by

$$P_1 := E \in \mathcal{L}(\mathcal{H}_n) \otimes \mathcal{L}(\mathcal{H}_n),$$
$$P_{\ell+1} := E \otimes (I - P_\ell) + (I - E) \otimes P_\ell, \text{ for } \ell \geq 1.$$

Bound Entangled Werner States

Define a family of projections $\{P_\ell\}$ of ever-increasing size recursively by

$$P_1 := E \in \mathcal{L}(\mathcal{H}_n) \otimes \mathcal{L}(\mathcal{H}_n),$$
$$P_{\ell+1} := E \otimes (I - P_\ell) + (I - E) \otimes P_\ell, \text{ for } \ell \geq 1.$$

- For any α in the "interesting region", P_ℓ is the projection onto the negative eigenspace of $(id_n \otimes T)(\rho_\alpha)^{\otimes \ell}$. Our tests for k-block positivity from earlier thus apply by looking at this family of projections.

Bound Entangled Werner States

In particular, in the $\alpha = \frac{2}{n}$ case, we can show that ρ_α is bound entangled if and only if $\lim_{\ell \to \infty} \left\| P_\ell \right\|_{S(2)} = \frac{1}{2}$.

Bound Entangled Werner States

In particular, in the $\alpha = \frac{2}{n}$ case, we can show that ρ_α is bound entangled if and only if $\lim_{\ell \to \infty} \left\| P_\ell \right\|_{S(2)} = \frac{1}{2}$.

So get calculating!

What We Have Seen

- The Schmidt operator norms are powerful tools for determining k-block positivity of Hermitian operators

What We Have Seen

- The Schmidt operator norms are powerful tools for determining k-block positivity of Hermitian operators
- They are *especially* useful for detecting k-block positivity of operators with two eigenvalues. In particular, they may be able to help solve the NPPT bound entanglement problem.

What We Have Seen

- The Schmidt operator norms are powerful tools for determining k-block positivity of Hermitian operators
- They are *especially* useful for detecting k-block positivity of operators with two eigenvalues. In particular, they may be able to help solve the NPPT bound entanglement problem.
- We have seen several ways to bound the Schmidt norms of projections. It is known that our bounds are asymptotically (in the dimension of the local Hilbert space) tight up to a universal multiplicative constant.

What We Would Like to See

- We would like a lower bound on the Schmidt norm of projections that is tight (not just asymptotically tight).

What We Would Like to See

- We would like a lower bound on the Schmidt norm of projections that is tight (not just asymptotically tight).
- More importantly, we would like a characterization of the projections that *attain* the tight lower bound. The family of Werner state projections that were discussed seem somehow "maximally entangled", so the (perhaps premature) conjecture is that they are among the projections that attain the lower bound.

Bibliography

- N. Johnston, D. W. Kribs, *Schmidt Norms for Quantum States*, preprint (2009). arXiv:0909.3907 [quant-ph]
- P. Gawron, Z. Puchala, J. A. Miszczak, L. Skowronek, M.-D. Choi, K. Zyczkowski, *Local numerical range: a versatile tool in the theory of quantum information*, preprint (2009). arXiv:0905.3646v1 [quant-ph]
- Ł. Skowronek, E. Størmer, and K. Życzkowski, *Cones of positive maps and their duality relations*. J. Math. Phys. **50**, 062106 (2009). arXiv:0902.4877v1 [quant-ph]