QUANTUM SEMIDEFINITE PROGRAMS

NATHANIEL JOHNSTON[1]

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In quantum information theory, semidefinite programs often depend on a Hermicity-preserving linear map $\Phi : \mathcal{L}(\mathcal{H}_n) \to \mathcal{L}(\mathcal{H}_m)$ as well as two Hermitian operators $A \in \mathcal{L}(\mathcal{H}_n)$ and $B \in \mathcal{L}(\mathcal{H}_m)$, and have the following primal and dual forms:

$$
\begin{array}{rll@{\qquad}rll}
 & \multicolumn{2}{c}{\textbf{Primal problem}} & & \multicolumn{2}{c}{\textbf{Dual problem}} \\
 & \text{maximize:} & \mathrm{Tr}(AP) & & \text{minimize:} & \mathrm{Tr}(BQ) \\
(1) & \text{subject to:} & \Phi(P) \leq B & & \text{subject to:} & \Phi^\dagger(Q) \geq A \\
 & & P \geq 0 & & & Q \geq 0
\end{array}
$$

However, the standard form for semidefinite programs depends on operators $D = D^*$ and $\{G_i\}_{i=1}^s \in \mathcal{L}(\mathcal{H}_r)$ and has the following primal and dual forms:

$$
\begin{array}{rll@{\qquad}rll}
 & \multicolumn{2}{c}{\textbf{Primal problem}} & & \multicolumn{2}{c}{\textbf{Dual problem}} \\
 & \text{maximize:} & x^* c & & \text{minimize:} & \mathrm{Tr}(DY) \\
(2) & \text{subject to:} & \sum_{i=1}^s x_i G_i \leq D & & \text{subject to:} & \mathrm{Tr}(G_i Y) = c_i \quad \forall\, 1 \leq i \leq s \\
 & & & & & Y \geq 0
\end{array}
$$

In this note I will show that the optimization problem (1) is truly a semidefinite program by converting it into the standard form (2). The interested reader can verify that the two forms are in fact equivalent.

Define a linear map $\Psi : \mathcal{L}(\mathcal{H}_n) \to (\mathcal{L}(\mathcal{H}_m) \oplus \mathcal{L}(\mathcal{H}_n))$ by

$$\Psi(P) := \begin{bmatrix} \Phi(P) & 0 \\ 0 & -P \end{bmatrix}.$$

Then the requirement that $\Phi(P) \leq B$ and $P \geq 0$ is equivalent to

$$\Psi(P) \leq \begin{bmatrix} B & 0 \\ 0 & 0 \end{bmatrix}.$$

The dual map $\Psi^\dagger : (\mathcal{L}(\mathcal{H}_m) \oplus \mathcal{L}(\mathcal{H}_n)) \to \mathcal{L}(\mathcal{H}_n)$ is given by

$$\Psi^\dagger\Big(\begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} \Big) = \Phi^\dagger(Q) - R.$$

Thus, we can write the semidefinite program (1) in the following form:

$$
\begin{array}{rll@{\qquad}rll}
 & \multicolumn{2}{c}{\textbf{Primal problem}} & & \multicolumn{2}{c}{\textbf{Dual problem}} \\
 & \text{maximize:} & \mathrm{Tr}(AP) & & \text{minimize:} & \mathrm{Tr}(DY) \\
(3) & \text{subject to:} & \Psi(P) \leq D & & \text{subject to:} & \Psi^\dagger(Y) = A \\
 & & & & & Y \geq 0
\end{array}
$$

where $D := \begin{bmatrix} B & 0 \\ 0 & 0 \end{bmatrix}$ and $Y := \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix}$. The inequality in the dual problem could be replaced by equality because of the flexibility of the arbitrary positive operator R. Now let $\{E_a\}, \{F_a\}$ be families of left and right generalized Choi-Kraus operators for Ψ. Denote the (k,l)-entry of P by p_{kl} and the (i,j)-entry of E_a or F_a by e_{aij} or f_{aij}, respectively. Then

$$\Psi(P) = \sum_a E_a P F_a = \sum_a (\sum_{kl} e_{aik} p_{kl} f_{alj})_{ij} = \sum_{kl} p_{kl} G_{kl},$$

where

$$G_{kl} := \sum_a \begin{bmatrix} e_{a1k} f_{al1} & e_{a1k} f_{al2} & \cdots & e_{a1k} f_{al(m+n)} \\ e_{a2k} f_{al1} & e_{a2k} f_{al2} & \cdots & e_{a2k} f_{al(m+n)} \\ \vdots & \vdots & \ddots & \vdots \\ e_{a(m+n)k} f_{al1} & e_{a(m+n)k} f_{al2} & \cdots & e_{a(m+n)k} f_{al(m+n)} \end{bmatrix}.$$

Finally, defining $x := \mathrm{vec}(P)$ and $c := \mathrm{vec}(A)$ (where vec refers to the vectorization of a matrix, which stacks each of its columns on top of each other) shows that the primal problem (1) is in the form of the primal problem (2). The dual problem can similarly be seen to be in standard form.

[1]Department of Mathematics & Statistics, University of Guelph, Guelph, ON, Canada N1G 2W1