LEMMA OF THE MONTH #4
NO SIMILARITY-INVARIANT MATRIX NORM

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A norm $\|\cdot\|$ on M_n is said to be *weakly unitarily-invariant* if $\|UXU^*\| = \|X\|$ for all $U, X \in M_n$ with U unitary. There are many weakly unitarily-invariant norms (such as the trace norm, operator norm, Frobenius norm, and numerical radius). As a natural strengthening of the concept of a weakly unitarily-invariant norm, one might wonder if there are any norms that are *similarity-invariant*; that is, that satisfy $\|SXS^{-1}\| = \|X\|$ for all $S, X \in M_n$ with S invertible. It turns out that no such norm exists, as the following lemma shows. This lemma appeared as Exercise IV.4.1 in [1].

Lemma 1 (No Similarity-Invariant Norm)**.** *Let $f : M_n \to \mathbb{R}$ be a function satisfying $f(SXS^{-1}) = f(X)$ for all $S, X \in M_n$ with S invertible. Then f is not a norm.*

Proof. Assume that f is a norm satisfying the hypothesis. Let $a > 0$ be real and consider the following 2×2 matrices (the same technique works for $n > 2$):

$$N := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad S := \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}.$$

Then $SNS^{-1} = aN$. If f were a norm, it would follow that $af(N) = f(aN) = f(SNS^{-1}) = f(N)$, so $a = 1$. Since $a > 0$ was arbitrary, we have reached a contradiction. □

References

[1] R. Bhatia, *Matrix analysis*. Volume 169 of Graduate texts in mathematics (1997).

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