LEMMA OF THE MONTH #2
A BACKWARD TRIANGLE INEQUALITY FOR MATRICES

NATHANIEL JOHNSTON[1]

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Recall that one of the defining properties of a matrix norm $\|\cdot\|$ is that it satisfies the triangle inequality $\|A+B\| \leq \|A\| + \|B\|$. What can we say about generalizing the *backward* triangle inequality to matrix norms? One backward triangle inequality is trivially true: $\big|\|A\| - \|B\|\big| \leq \|A-B\|$. What happens though, if we swap the roles of the absolute value and the matrix norm on the left-hand side? That is, if we recall that $|A| := \sqrt{A^*A}$, then can we say that $\big\||A| - |B|\big\| \leq \|A-B\|$? Not quite, but we can get close with the Frobenius norm, which we will denote by $\|\cdot\|_2$.

Theorem 1 (Araki-Yamagami [1])**.** *Let $A, B \in M_n$. Then*

$$\big\||A| - |B|\big\|_2 < \sqrt{2}\|A-B\|_2$$

The proof can be built up to rather simply from some lemmas involving commutant-type matrix relations, one of which I will present and prove here.

Lemma 2. *Let $A \in M_n$ be positive semi-definite and let $X \in M_n$ be arbitrary. Then $\|AX - XA\|_2 \leq \|AX + XA\|_2$.*

Proof. Use the Spectral Decomposition Theorem to write $A = UDU^*$, where U is unitary and $D \geq 0$ is diagonal, and let $Y = U^*XU$. Then it follows from unitary invariance of the Frobenius norm that

$$\|AX - XA\|_2^2 = \|DY - YD\|_2^2 \quad \text{and} \quad \|AX + XA\|_2^2 = \|DY + YD\|_2^2.$$

By focusing on the terms involving D and Y, writing the i^{th} entry of D as d_i, and writing $Y = (y_{ij})$, we see that these two values equal

$$\sum_{i,j=1}^{n} (y_{ij}(d_i - d_j))^2 \quad \text{and} \quad \sum_{i,j=1}^{n} (y_{ij}(d_i + d_j))^2,$$

respectively. Because A is positive semi-definite, each d_i is non-negative. It follows that each term in the sum on the left is less than or equal to the corresponding term in the sum on the right, and the result follows. □

Somewhat surprisingly, this lemma has a simple corollary that appears to be, upon first glance, a considerably stronger statement. Its proof is left as an exercise for the interested reader *[Hint: Consider 2×2 block matrices]*.

Corollary 3. *Let $A, B \in M_n$ be positive semi-definite and let $X \in M_n$ be arbitrary. Then $\|AX - XB\|_2 \leq \|AX + XB\|_2$.*

References

[1] H. Araki and S. Yamagami, *An inequality for the Hilbert-Schmidt norm*, Commun. Math. Phys., **81** (1981) 89–98.

[1] Department of Mathematics & Statistics, University of Guelph, Guelph, ON, Canada N1G 2W1