LEMMA OF THE MONTH #1
UNITAL CHANNEL EIGENVALUE MAJORIZATION

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The following lemma makes precise something that feels quite natural when thought about physically: a unital channel (that is, a completely positive, trace-preserving map $\mathcal{E}$ for which $\mathcal{E}(I) = I$) can only increase the impurity (or mixedness) of quantum states. It has several simple consequences that are of great use when dealing with unital channels, and furthermore its proof makes excellent use of classical machinery. It was originally due to Uhlmann [1, 2], but has recently appeared in [3]. The proof provided here is from the latter source.

Lemma 1. *Suppose $\rho = \mathcal{E}(\sigma)$ for a unital channel $\mathcal{E}$. Then the ordered spectrum r of ρ is majorised by the ordered spectrum s of σ.*

Proof. Let p_k and $|e_k\rangle$ (respectively, q_k and $|f_k\rangle$) denote the k^{th} eigenvalue and normalized eigenvector of ρ (respectively, σ). Clearly, $q_k = \sum_{k'} D_{kk'} p_{k'}$, where $D_{kk'} := \mathrm{Tr}(|f_k\rangle\langle f_k|\mathcal{E}(|e_{k'}\rangle\langle e_{k'}|))$. From the fact that $\mathcal{E}$ is trace-preserving one infers that $\sum_k D_{kk'} = 1$, while from the fact that $\mathcal{E}$ is unital one infers that $\sum_{k'} D_{kk'} = 1$. It follows that the matrix $D := (Dkk')$ is bistochastic. Thus, by the Hardy-Littlewood-Polya theorem, we have that r is majorized by s. □

Among others, this lemma admits the following simple corollary, whose proof is left as an exercise to the interested reader.

Corollary 2. *For a unital quantum channel $\mathcal{E}$ acting on a positive operator ρ, we have that $\mathrm{rank}(\mathcal{E}(\rho)) \geq \mathrm{rank}(\rho)$.*

References

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[2] I. Bengtsson, and K. Życzkowski, *Geometry of quantum states*, Cambridge University Press (2006).

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